CS 710: Complexity Theory	Date: Feb. 8th, 2024

Lecture 6: Random Walk for USTCONN (Continued)

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1 Outline

Remember that last lecture's scribe, we have left two question:

- is $\theta_{ijuv} = 1, \forall i, j, u, v$ in a one-dimensional finite chain?
- does this hold for all graphs?

We will show that $\theta_{ijuv} =$ constant ≤ 1 for any given graph $G = (V, E)$, and this along with the Markov's inequality give a one-sided any small ε-error randomized log space algorithm for the problem USTCONN.

2 General Case for Random Walk on ij-*commute* Path

Theorem 1. $\theta_{ijuv} = \theta_{ijuv'}$

Proof. By definition,

$$\begin{aligned}\theta_{ijuv} &= \sum_k \mathbb{E}[Y_k = u \wedge Y_{k+1} = v] \\ &= \sum_k Pr[Y_k = u] \cdot Pr[Y_{k+1} = v | Y_k = u] \\ &= \sum_k Pr[Y_k = u] \cdot \frac{1}{\deg(u)}\end{aligned}$$

we see that this formula is irrelevant to the vertex v, so we prove it. □

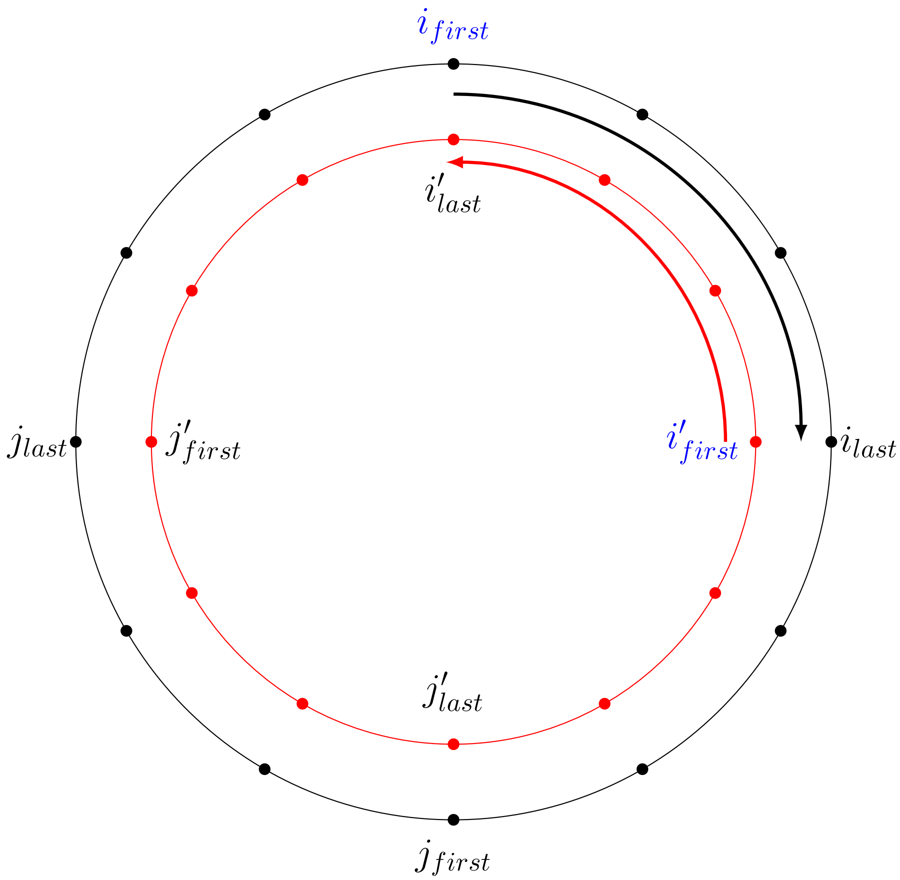
Lemma 1. There exists an *ij-commute* path $\mathcal{C}^R$ that rearrange the random variables in the *ij-commute* path $\mathcal{C}$ such that all random variables have a one-to-one mapping and the transition is valid.

Proof. The outer circle in the graph shows the original path C, where it starts from node i_{first}, all the way through node i_{last} (possibly $i_{first} = i_{last}$), and there is no existence of node i afterwards until reaching the terminal point i_{first} again. The similar definitions for j_{first} and j_{last} hold.

Now the amazing construction is that let us define C^R to be a one-to-one mapping for C, except that

- the nodes being visited in C are now arranged in the reverse direction
- the starting node become $i'_{first} = i_{last}$

Just as shown in the graph, we can see that amazingly C^R also follows the *ij-commute* property.



□

Theorem 2. $\theta_{ijuv} = \theta_{ijvu}$

Proof. The probability to get the path $\mathcal{C}^R$ is the same as $\mathcal{C}$, since

$$Pr[\mathcal{C}] = \prod_i Pr[Y_{i+1}|Y_i] = \prod_i \frac{1}{\deg(Y_i)} = Pr[\mathcal{C}^R]$$

We also have the number of occurrence of $count(\{u,v\})$ the same as $count(\{v,u\})$ in the same path $\mathcal{C}$, and hence the same for both $\mathcal{C}$ and $\mathcal{C}^R$.

This leads to the conclusion since

$$\theta_{ijuv} = \sum_{\mathcal{C}} Pr[\mathcal{C}] \cdot count(\{u, v\})$$
$$\theta_{ijvu} = \sum_{\mathcal{C}^R} Pr[\mathcal{C}^R] \cdot count(\{u, v\})$$

□

Theorem 3. $\theta_{ijuv} = \theta_{ijij} = \text{constant} \leq 1$

Proof. By applying the theorem above alternatively, we directly have $\theta_{ijuv} = \theta_{ijij} = \theta_{iju'v'} =$ constant.

For any graph G, we have $\theta_{ijij} \leq 1$ with similar proof as in last lecture on a one-directional semi-finite chain. □

We can see the proofs are so charming as the core of the proof is simply the rearrangement of all nodes in a random walk, but still preserves the *ij-commute* property.

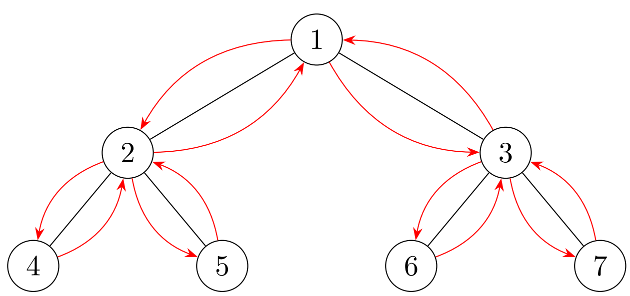
3 USTCONN

Lemma 2. The expected length of random walk $\mathbb{E}[W] \leq 2|E|$

Proof. Since each edge may have a chance to be traversed, we can relax $\mathbb{E}[W] = \sum_{(u,v)\in E} \theta_{ijuv} + \theta ijvu \leq 2|E|$ □

Theorem 4. For a random walk path $\mathcal{C}$ in an undirected graph $G = (V, E)$, the expected length to cover all the nodes at least once should no be no greater than $4|V| \cdot |E|$.

Proof. For an undirected graph $G = (V, E)$, we can have a spanning tree T, and performing a depth-first-search as shown in the figure, but with random walk.



Therefore, we know that the expected number of steps T to perform such search equals to the summation of the expected number of steps T_{ij} for all i, j on the path of such traversal. While $T_{ij} < \theta_{ijij}$, we have $T < 2|V| \cdot \mathbb{E}[W] < 2|V| \cdot 2|E| = 4|V| \cdot |E|$. □

Definition 1. *USTCONN* is a GAP but on an undirected graph.

Theorem 5. USTCONN can be solved with at most $\left(\frac{1}{2}\right)^n$ one-sided error by running a random walk for n times, with at least length $8|V| \cdot |E|$, starting from node s.

Proof. By Markov's inequality saying that

$$Pr[X \geq a] \geq \frac{\mathbb{E}[X \geq a]}{a}$$

we have $Pr[W \geq 8|V| \cdot |E|] \geq \frac{1}{2}$.

This implies that if we run a random walk with length no smaller than $8|V| \cdot |E|$, we have more than half the chance to cover all the nodes, and hence we have at most half the chance making a false negative statement (falsely claiming that there does not exist a path from s to t).

By applying this algorithm n time, we get the conclusion. □

Corollary 1. USTCONN can be solved with randomized algorithm with arbitrarily small one-sided error in log space complexity.

Proof. Since during the random walk we don't need to record any path, but simply record several pivot nodes, including current node, s and t, we can solve it in log space. □

Now, we can somehow answer the question raised in last lecture. Compared to GAP, USTCONN is expected to be solved more efficiently. As we know GAP can be solved in log space non-deterministically, now we somehow reach a better space complexity, that is USTCONN can be solved in log space randomly.

There exists a derandomization method that can solve USTCONN in log space deterministically, which may be covered in the future lectures. Overall, we can see that for this particular problem, an undirected graph has a lower space complexity than directed graph assuming NL $\neq$ L.